Lecture 20

AoI and Sampling

Reading: Wait or Update TIT 2017.

JSAC AoI survey

Sun, Cyr 2019.

Ornee, Sun 2020

The problem is reformulated as.

$$\bar{P}_{opt} = \inf_{\pi \in \Pi_1} \lim_{i\to\infty} \frac{\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})]}{\sum_{j=1}^{i} E[Y_j + Z_j]} . \qquad (1)$$

Step 3.

Consider the following problem:

$$h(c) = \inf_{\pi \in \Pi_1} \lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i-1} E\left[q(Y_j, Z_j, Y_{j+1}) - c(Y_j + Z_j)\right] . \qquad (2) .$$

Lemma (a) $\bar{P}_{opt} \gtreqless c$ if and only if $h(c) \gtreqless 0$.

(b) If $h(c) = 0$, then the solutions to (1) and (2) are identical.

Proof. "$\Rightarrow$" if $\bar{P}_{opt} \le c$, then for any $\varepsilon > 0$.

there exist a policy $\pi = (Z_1, Z_2 \cdots)$ satisfying.

$$\lim_{i\to\infty} \frac{\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})]}{\sum_{j=1}^{i} E[Y_j + Z_j]} \le c + \varepsilon .$$

$$\therefore \lim_{i\to\infty} \frac{\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})] - c \sum_{j=1}^{i} E(Y_j + Z_j)}{\sum_{j=1}^{i} E[Y_j + Z_j]} \le \varepsilon .$$

$$E[Y_j] > 0 ,$$

$$\frac{1}{i} \sum_{j=1}^{i} E(Y_j + Z_j) \ge m > 0 .$$

$$\lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i} E[Y_j + Z_j] \text{ exist.}$$

$$\lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i} E[q_r(Y_j, Z_j, Y_{j+1})] - c\frac{1}{i} \sum_{j=1}^{i} E[Y_j+Z_j]$$

$$\leq \varepsilon \frac{1}{i} \sum_{j=1}^{i} E[Y_j+Z_j]$$

if $\lim_{i\to\infty} a_i = a$. $\lim_{i\to\infty} b_i = b \neq 0$, then

$$\lim_{i\to\infty} \frac{a_i}{b_i} = \frac{a}{b}. \qquad \lim_{i\to\infty} a_i b_i = ab.$$

The choice of $\varepsilon > 0$ is arbitrary

$$\inf\left\{\lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i} E[q_r(Y_j, Z_j, Y_{j+1})] - c\frac{1}{i} \sum_{j=1}^{i} E[Y_j+Z_j]\right\}$$

$$\leq 0.$$

$$h(c) \leq 0.$$

"$\Leftarrow$" if $h(c) \leq 0$. then $\bar{P}_{opt} \leq c$.

if $h(c) \leq 0$, then for any $\varepsilon > 0$.

there exists a policy $\pi = (z_1, z_2, \cdots)$ satisfying

$$\lim_{i\to\infty}\left[\frac{1}{i}\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})] - c\frac{1}{i}\sum_{j=1}^{i} E(Y_j + Z_j)\right]$$

$$\leq \varepsilon \lim_{i\to\infty} \frac{1}{i}\sum_{j=1}^{i} E(Y_j + Z_j)$$

$$\lim_{i\to\infty} \frac{\frac{1}{i}\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})]}{\frac{1}{i}\sum_{j=1}^{i} E(Y_j + Z_j)} \leq c + \varepsilon.$$

The choice of $\varepsilon > 0$ is arbitrary.

$\varepsilon = 1$. $\qquad \pi_1 \in \Pi_1$

$\varepsilon = \frac{1}{2}$. $\qquad \pi_2 \in \Pi_1$

$\varepsilon = \frac{1}{3}$ $\qquad \vdots$

$\vdots \qquad \pi_n \in \Pi_1$

$\vdots$

$$\inf_{\pi \in \{\pi_1, \pi_2, \cdots\}} \lim_{i\to\infty} \frac{\frac{1}{i}\sum_{j=1}^{i} E[q(Y_j, Z_j, Y_{j+1})]}{\frac{1}{i}\sum_{j=1}^{i} E(Y_j + Z_j)} \leq c.$$

$\bar{P}_{opt} \leq c$.

Next we need

$\bar{P}_{opt} < c$ if & only if $h(c) < 0$.

$\bar{P}_{opt} \geqslant c$ if & only if $h(c) \geqslant 0$.

By this,

" $\bar{P}_{opt} \begin{matrix} < \\ = \\ > \end{matrix} c$ if & only if $h(c) \begin{matrix} < \\ = \\ > \end{matrix} 0$. ".

Proof of Part (b) is omitted. $\Box$.

we only need to solve.

$$h(c) = \inf_{\pi \in \Pi_1} \lim_{i \to \infty} \frac{1}{i} \sum_{j=1}^{i-1} E\left[q(Y_j, Z_j, Y_{j+1}) - c(Y_j + Z_j)\right]. \quad (2)$$

and seek $\bar{P}_{opt}$ such that

$$h(\bar{P}_{opt}) = 0.$$

$$h(\bar{P}_{opt}) = \inf_{\pi \in \Pi_1} \lim_{i \to \infty} \frac{1}{i} \sum_{j=1}^{i-1} E\left[q(Y_j, Z_j, Y_{j+1}) - \bar{P}_{opt}(Y_j + Z_j)\right]. \quad (3).$$

Step 4. Problem decomposition:

We want to show that

$$h(\bar{P}_{opt}) = \lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i-1} \inf_{Z_j \geq 0} E[q(Y_j, Z_j, Y_{j+1}) - \bar{P}_{opt}(Y_j + Z_j)],$$

Z_j depends on $(Y_1, \cdots, Y_j, Z_1, \cdots Z_{j-1})$.

decoupled optimization problems:

$$h(\bar{P}_{opt}) = \lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i-1} \inf_{Z_j \geq 0} E[q(Y_j, Z_j, Y_{j+1}) - \bar{P}_{opt}(Y_j + Z_j)],$$

Z_j depends on Y_j.

a series of per-sample optimization problems.

sufficient statistics.

$$\min_x E[f(x, Y, z)] \quad (4)$$

if $E[f(x, Y, z)] = E[g(x, Y)]$ then (4) is equivalent to

$$\min_x E[g(x, Y)].$$

then we only need to use Y for solving (4).

Lemma: Y_j is a sufficient statistic for optimizing Z_j in (3).

Proof: the Y_j's are i.i.d.

In (3). Z_j only occurs in the follow term.

$$E[q_r(Y_j, Z_j, Y_{j+1}) - \bar{P}_{opt} Z_j],$$

This term depends on $(Y_1, \cdots, Y_j, Z_1, \cdots Z_{j-1})$ only through Y_j.

$\therefore$ Y_j is a sufficient statistic for optimizing Z_j in (3) $\blacksquare$.

Read: Lemma 4.

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$$\inf_{Z_j \geq 0} E[q_r(Y_j, Z_j, Y_{j+1}) - \bar{P}_{opt} Z_j] \quad (5).$$

$Pr(Z_j \in A \mid Y_j = y)$.

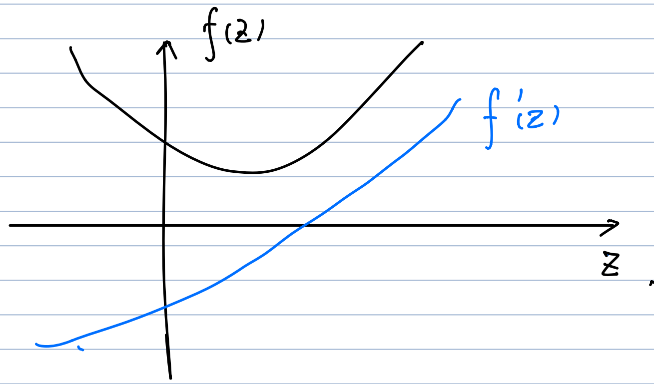
Consider the following problem.

$$\inf_{z \geq 0} E[q_r(y, z, Y_{j+1})] - \bar{P}_{opt} z \quad (6).$$

Recall $q_r(y, z, y') = \int_y^{y+z+y'} P(\tau)\, d\tau.$

$P(t)$ is non-decreasing.

$\frac{\partial q_r(y, z, y')}{\partial z} = P(y+z+y')$, which is non-dcreasing in z.



$q(y, z, y')$ is convex in z.

$E[q(y, z, Y_{i+1})]$ is convex in z.

$\therefore$ (6) is convex optimization problem.

Lemma:

$z_{min}(y) = \inf\{t \geq 0 : E[p(y+t+Y_{j+1})] \geq \bar{p}_{opt}\}$.

$z_{max}(y) = \inf\{t \geq 0 : E[p(y+t+Y_{j+1})] > \bar{p}_{opt}\}$.

z is an optimal solution to (6) if & only if

$z \in [z_{min}(y), z_{max}(y)]$.

Proof: Lemma 5.

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The set of optimal solutions to (5) is

$\{Pr(Z_j \in A \mid Y_j = y) : Z_j \in [z_{min}(y), z_{max}(y)] \text{ with prob } 1\}$.

Pick one of the optimal solutions to (5).

$Z_j = z_{min}(Y_j) \quad \forall j$.

$= \inf\{t \geq 0 : E[p(Y_j + t + Y_{j+1}) | Y_j] \geq \bar{p}_{opt}\}$.

because Y_j are i.i.d., the $\{Z_j\}$'s are i.i.d.

$$h(\bar{p}_{opt}) = \lim_{i\to\infty} \frac{1}{i} \sum_{j=1}^{i-1} \inf_{\substack{Z_j \geq 0, \\ Z_j \text{ depends on } Y_j}} E[q(Y_j, Z_j, Y_{j+1}) - \bar{p}_{opt}(Y_j + Z_j)].$$

$$= \inf_{\substack{Z_j \geq 0 \\ Z_j \text{ depend } Y_j}} E[q(Y_j, Z_j, Y_{j+1}) - \bar{p}_{opt}(Y_j + Z_j)]$$

$= 0$.

$S_{i+1}(\beta) = \inf\{t \geq D_i(\beta) : E[p(\Delta(t + Y_{i+1})) | Y_i] \geq \beta\}$.

where $D_i(\beta) = S_i(\beta) + Y_i$, $\Delta(t) = t - S_i(\beta)$,

β is the root of.

$$E\left[\int_{D_i(\beta)}^{D_{i+1}(\beta)} p(\Delta(t))\, dt\right] - \beta E[D_{i+1}(\beta) - D_i(\beta)] = 0 \quad (7).$$

and $\beta = \bar{p}_{opt}$.

Reading. Sun. Cyr 2019.

Key ideas:

1. fractional programing,
2. sufficient statistic,
3. convex optimization, prob theory.
4. (7) has a unique root,